§14.1: Multivariable Functions

09/27/21

Definition: A multivariable function (of n-variables w/ real values) is a function $f: D \subseteq \mathbb{R}^{n} \to \mathbb{R}$

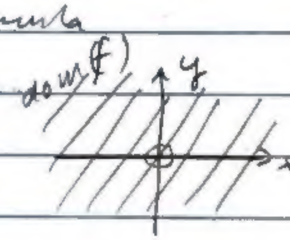
$\uparrow$ function's name $\uparrow$ function's domain $n \leftarrow$ # of variables $\uparrow$ output real #

$dom(f)$ = domain of f

$ran(f) = \{ f(\vec{x}) : \vec{x} \in dom(f) \}$

NB: Often, we won't explicitly state the domain of a function given formulaically. We'll use "the natural domain" in that cases, i.e. the set of all inputs w/ defined outputs given by the formula

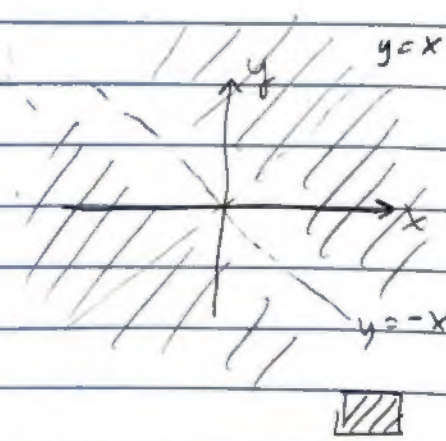
Ex: $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$



$dom(f) = \{ (x,y) \in \mathbb{R}^2 : \dfrac{x^2-y^2}{x^2+y^2} \text{ is defined} \}$

$= \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0 \} = \{ (x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0) \}$

Ex: $f(x,y) = \dfrac{x^2+y^2}{x^2-y^2}$



$dom(f) = \{ (x,y) : \dfrac{x^2+y^2}{x^2-y^2} \text{ is defined} \}$

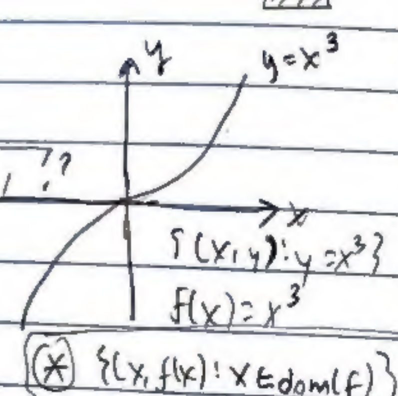
$= \{ (x,y) \in \mathbb{R}^2 : x^2-y^2 \neq 0 \}$

$= \{ (x,y) : x \neq \pm y \}$

$= \{ (x,y) \in \mathbb{R}^2 : |x| \neq |y| \}$

Definition: The graph of a function f is

$graph(f) = \{ (\vec{x}, f(\vec{x})) : \vec{x} \in dom(f) \}$



$\circledast$ $\{ (x, f(x)) : x \in dom(f) \}$

Ex: What is the shape of $f(x,y) = \sqrt{x^2+y^2+1}$??

Sol: Setting $z = f(x,y)$

$z = \sqrt{x^2+y^2+1}$, i.e. $z^2 = x^2+y^2+1$ & $z \geq 0$

i.e. $-x^2-y^2+z^2=1$ & $z \geq 0$

$\uparrow$ two-sheet hyperboloid

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i.e. $-x^2-y^2+z^2=1$ & $z \geq 0$

↑ two-sheet hyperboloid

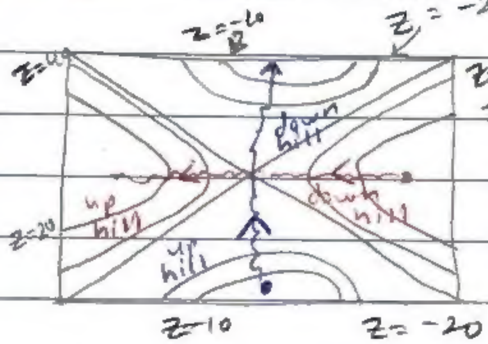
∴ graph(f) is the upper sheet of a two-sheet hyperboloid ▨

Question: How can we represent a two-variable function in $\mathbb{R}$-space?

Answer: Build a contour map (aka level curves, or elevation map)

Ex:

hyperbolic paraboloid →



← contour map of hyperbolic paraboloid

Ex: The unit hypersphere is:

$S^3 = \{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2+z^2+t^2=1\}$ ↖ $|t| \leq 1$

The t-level sets look like:

$t=-1$:  ← starts w/ point

sphere getting bigger

$t=-\frac{1}{2}$: 

$t=0$: 

$t=\frac{1}{2}$  getting smaller

smaller

$t=1$ 

Notation:

$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$

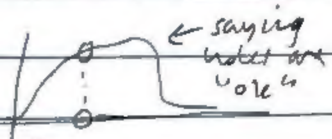
or

$f(\vec{x}) \to L$ as $\vec{x}\to\vec{a}$

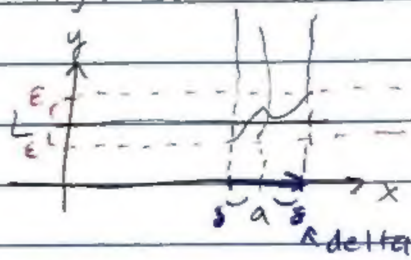
§14.2: Limits & Continuity:

In calculus II, the formal definition of limits is:

~~Definition: Let f be a function & let "a" be a limit point of dom(f)~~

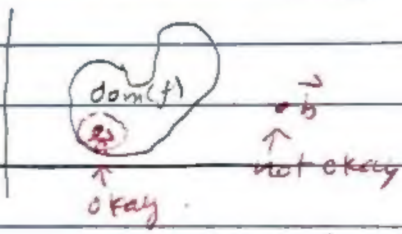


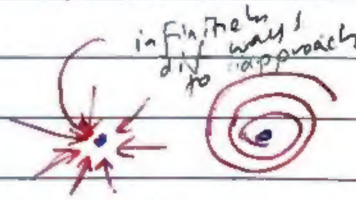
Definition: Let f be a multivariable function & let $\vec{a} \in \mathbb{R}^n$ be a limit point of dom(f). The limit as $\vec{x}$ tends to $\vec{a}$ of f, is $L \in \mathbb{R}$ when for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\vec{a} \neq \vec{x} \in$ dom(f) we have $|\vec{x} - \vec{a}| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$



"calculus III version of "one-sided limits are equal""

Proposition (Curves Criterion): Let f be a function & $\vec{a}$ a limit point @ its domain. $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ if & only if for all curves $\vec{r}(t)$ in dom(f) such that (s.t.) $\lim_{t\to 0^+} \vec{r}(t) = \vec{a}$ we have $\lim_{t\to 0^+} f(\vec{r}(t)) = L$.





Ex: Show $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist

Solution: Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ & $l_{ab}(t) = \langle at, bt \rangle$

Note that $\lim_{t\to 0} l_{a,b}(t) = \langle 0, 0 \rangle$

for all $t \neq 0$, we have $f(l_{a,b}(t)) = \frac{(at)^2 - (bt)^2}{(at)^2 + (bt)^2}$

$= \frac{(a^2-b^2)t^2}{(a^2+b^2)t^2} = \frac{a^2-b^2}{a^2+b^2}$ $\therefore \lim_{t\to 0} f(l_{0,1}(t)) = \lim_{t\to 0} \frac{a^2-b^2}{a^2+b^2}\Big|_{a=0,\, b=1}$

$= \frac{0-1}{0+1} = -1$, check $\lim_{t\to 0} f(l_{1,1}(t)) = 0 \neq -1$ $\therefore$ by the curves criterion $\lim_{\vec{x}\to 0} f(\vec{x})$ does not exist. $\boxed{}$